Equilibrium in a complete-market endowment economy

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1. Introduction

This lecture describes equilibria for a pure exchange infinite horizon economy with stochastic endowments. This kind of economy is useful for studying risk sharing, asset pricing, and consumption. We describe two market structures. Both are referred to as complete markets economies. They allow more comprehensive sharing of risks than do the incomplete markets economies, or economies with information- or enforcement- frictions.

This note is based largely on Stokey, Lucas, and Prescott, 1989.

2. Physical settings

There are I agents named $1, 2, \cdots I$. Agent i owns a stochastic endowment of one good $y_t^i(s^t)$ that depends on the history of a stochastic process $s^t = \{s_0, s_1, \cdots s_t\}$. The stochastic process s_t (not necessarily Markov) is *global* and publicly observable and is common to everyone. Household i chooses a consumption policy c_t^i ($c_t^i(s^t)$ is also history-dependent) in order to maximize its utility

$$U^i = \max_{\{c_t^i\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta_i^t u_i(c_t^i)$$

In this note, we adopt an assumption widely employed in macroeconomics that $\beta_i = \beta$.

Suppose that there exists a market where agents can trade claims to consumption. We consider two market structures:

1. Arrow-Debreu structure with complete market in dated contingent claims
   Here markets meet at time 0 to trade claims to consumption at all times $t > 0$ that are contingent on all possible histories up to t.
2. Sequential trading structure with complete one-period Arrow securities
   In this economy, agents trade only one-period-ahead state-contingent claims. Here trades occur at each date $t \geq 0$. Trades for history s_{t+1}–contingent date-$t+1$ goods occur only at the particular date t history s^t that has been reached at t.

These two market structure entail different assets and timing of trades. As we will show, they have identical equilibrium consumption allocations. Furthermore, although the endowment depends on history s^t, after trading the equilibrium consumption allocation at time t depends only on the aggregate endowment realization at time t and initial distribution of wealth. The information and enforcement frictions will break this result.

If trading is allowed, a feasible consumption allocation should satisfy

$$\sum_{i=1}^{I} c_t^i(s^t) \leq \sum_{i=1}^{I} y_t^i(s^t), \quad \forall s^t$$

3. Pareto optimality of allocation

An allocation is said to be efficient if it is Pareto optimal: it has the property that any reallocation that makes one agent strictly better off also makes one or more other agents worse off.

To find the optimal consumption allocation of the economy in the previous section, imagine a fictitious social planner. The planner chooses consumption allocation $c^i = \{c_t^i\}_{t=0}^{\infty}$, $i = 1, 2, \cdots, I$ to maximize

$$\sum_{i=1}^{I} \theta_i U^i(c^i),$$

subject to $\sum_{i=1}^{I} c_t^i(s^t) \leq \sum_{i=1}^{I} y_t^i(s^t)$. If the consumption allocation c^i solves the planner's problem for a set of nonnegative θ_i's, then the allocation is Pareto optimal. The Lagrangian of the planner's maximization problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{s^t} \left[\sum_{i=1}^{I} \theta_i \beta^t u_i(c_t^i(s^t)) \pi_t(s^t) + \lambda_t(s^t) \sum_{i=1}^{I} (y_t^i(s^t) - c_t^i(s^t)) \right].$$

where $\pi_t(s^t)$ is the unconditional probability of observing a particular sequence of events s^t.

The first order condition with respect to $c_t^i(s^t)$ is given by

$$\theta_i \beta^t u_i'(c_t^i(s^t)) \pi_t(s^t) = \lambda_t(s^t),$$

which implies

$$\frac{u_i'(c_t^i(s^t))}{u_1'(c_t^1(s^t))} = \frac{\theta_1}{\theta_i}.$$

As a result, $c_t^i(s^t) = u_i'^{-1}[\theta_i^{-1} \theta_1 u_1'(c_t^1(s^t))]$. Substituting $c_t^i(s^t)$ into the constraint gives

$$\sum_{i=1}^{I} u_i'^{-1}[\theta_i^{-1} \theta_1 u_1'(c_t^1(s^t))] = \sum_{i=1}^{I} y_t^i(s^t),$$

with which we can solve for $c_t^1(s^t)$. Therefore, given θ_i, $c_t^i(s^t)$ depends only on the current realization of the aggregate endowment, not on the history s^t or the distribution of individual endowments realized at t.

Proposition 1. *An efficient allocation is a function of the realized aggregate endowment and depends neither on the specific history leading up to that outcome nor on the realizations of individual* *endowments;* $c_i^t(s^t) = c_i^\tau(\bar{s}^\tau)$ *for* s^t *and* $\bar{s}$ *such that* $\sum_j y_j^t(s^t) = \sum_j y_j(\bar{s})$.

4. Time 0 Trading

We now describe how an optimal allocation can be attained by a competitive equilibrium in the Arrow-Debreu market. Households trade dated history-contingent claims to consumption. All trades occur at time 0 after observing s_0. After time 0 , trades that were agreed to at time 0 are executed, but no more trades occur.

4.1. Complete market

In complete market, there is negligible transaction cost and perfect information:

- There is a price for every asset in every possible state of the world.
- A state-contingent claim can always be decomposed as a linear combination of Arrow securities.

Let $q_t^0(s^t)$ denote the price of Arrow security that pays one consumption good if the state is s^t and 0 otherwise. $q_t^0(s^t)$ is also known as *state prices*. An asset claim can be broken into a collection of Arrow securities. Assume that $q_t^0(s^t)$ has already been priced in the market. Then this asset is called redundant asset. Let $\{d_t(s^t)\}$ be a stream of claims on s^t-consumption good. The price of this asset is given by

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d_t(s^t).$$

For example

- *riskless consol*: $d_t(s^t) = 1$, then the price of this asset is

$$\sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t).$$

- *riskless strips*: $d_t(s^t) = 1$ if $t = \tau$ and 0 otherwise. Then the price of this asset is

$$\sum_{s^\tau} q_\tau^0(s^\tau).$$

- *Tail asset*: The value of an asset for a particular realization of s^τ. Let $p_\tau^0(s^\tau)$ be the time 0 price (in unit of s_0 consumption good) of an asset that entitles the owner to dividend stream $\{d_t(s^t)\}_{t \geq \tau}$ if history s^τ is realized,

$$p_\tau^0(s^\tau) = \sum_{t=\tau}^{\infty} \sum_{s^t | s^\tau} q_t^0(s^t) d_t(s^t).$$

Normally, we convert the price unit to the unit of time τ, history s^τ consumption good

$$p_\tau^\tau(s^\tau) = \sum_{t=\tau}^{\infty} \sum_{s^t|s^\tau} \frac{q_t^0(s^t)}{q_\tau^0(s^\tau)} d_t(s^t)) = \sum_{t=\tau}^{\infty} \sum_{s^t|s^\tau} q_t^\tau(s^t) d_t(s^t).$$

Here, $q_t^\tau(s^t)$ is the price of one unit of consumption delivered at time t, history s^t in terms of the date τ, history s^τ consumption good.

$q_t^0(s^t)$ is determined in the equilibrium of the market.

5. Competitive equilibrium

The household's problem

$$\begin{aligned}
&\max_{c_t^i} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u_i(c_t^i) \\
&s.t. \text{(Budget Constraint)} \quad \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) c_t^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) y_t^i(s^t) \quad \forall i \qquad (1) \\
&\text{(Market Clearing)} \quad \sum_{i=1}^{I} c_t^i(s^t) = \sum_{i=1}^{I} y_t^i(s^t) \quad \forall s^t.
\end{aligned}$$

Because the units of the price system are arbitrary, one of the prices can be normalized at any positive value. We shall set $q_0^0(s_0) = 1$, putting the price system in units of time 0 goods.

First order condition:

$$[c_t^i(s^t)] : \quad \beta^t u_i'(c_t^i(s^t)) \pi_t(s^t) = \mu_i q_t^0(s^t),$$

which implies that for all (i, j)

$$\frac{u_i'(c_t^i(s^t))}{u_j'(c_t^j(s^t))} = \frac{\mu_i}{\mu_j}.$$

Thus, ratios of marginal utilities between pairs of agents are constant across all histories and dates. As a result, the competitive equilibrium is Pareto optimal. The associated Pareto weights are $\theta_i = \mu_i^{-1}$. The shadow prices $\lambda_t(s^t)$ of the planner's problem is equal to Arrow-Debreu prices $q_t^0(s^t)$. That the allocations for the planner's problem and the competitive equilibrium are aligned reflects the two fundamental theorems of welfare economics. (See Green, Mas-Colell, and Whinston., 1995)

Example 5.1. *Risk sharing*

In this example, we consider the CRRA utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0.$$

The first order condition becomes

$$\beta^t c_t^i(s^t)^{-\gamma}\pi_t(s^t) = \mu_i q_t^0(s^t).$$

As a result,

$$\frac{c_t^i(s^t)}{c_t^j(s^t)} = \left(\frac{\mu_i}{\mu_j}\right)^{-1/\gamma}.$$

The special result from the CRRA utility is that consumption allocations to distinct agents are constant fractions of one another. Thus, there is extensive cross-history and cross-time consumption smoothing.

We could set $c_t^i(s^t) = \alpha_i \sum_j y_t^j(s^t)$, *where* α_i *is agent* i*'s fixed consumption share of the aggregate endowment. The equilibrium price is then*

$$q_t^0(s^t) = \mu_i^{-1}\alpha_i^{-\gamma}\beta^t \left(\sum_j y_t^j(s^t)\right)^{-\gamma} \pi_t(s^t).$$

α_i *can be found by using the budget constraint.*

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)\alpha_i \sum_j y_t^j(s^t) = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t) y_t^i(s^t)$$

5.1. Equilibrium computation - Negishi algorithm

1. Fix a positive value for one μ_i, say μ_1 , throughout the algorithm. Guess some positive values for the remaining μ_i 's. Then solve equations

$$c_t^i(s^t) = u_i'^{-1}[u_1'(c_t^1(s^t))\frac{\mu_i}{\mu_1}], \quad \sum_{i=1}^{I} u_i'^{-1}[\theta_i^{-1}\theta_1 u_1'(c_t^1(s^t))] = \sum_{i=1}^{I} y_t^i(s^t),$$

   for $c_i(s^t)$, $i = 1, 2, ..., I$.
2. Solve for the price $q_t^0(s^t)$.
3. For $i = 1, 2, ..., I$, check the budget constraint. For those i's for which the cost of consumption exceeds the value of their endowment, raise μ_i , while for those i's for which the reverse inequality holds, lower μ_i.
4. Iterate 1-3 until convergence.

Multiplying all of the μ_i's by a positive scalar amounts simply to a change in units of the price system. That is why we are free to normalize as we have in step 1.

6. Sequential trading

The Arrow-Debreu market structure is an idealized theoretical benchmark, which cannot capture the trading of financial assets such as stocks and bonds in actual economies. In this section, we build on the insight that one-period securities are enough to implement complete market. We describe a competitive equilibrium of this sequential-trading economy. With a full array of these one-period-ahead claims, the sequential-trading arrangement attains the same allocation as the competitive equilibrium that we described earlier.

6.1. Endogenous state variable

A key step in finding a sequential-trading arrangement is to identify a variable to serve as the state in a value function for the household at date t. The insight here is that in the time-0 trading economy, households hold a net claim to delivery of goods in the future,

$$W_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau)[c_\tau^i(s^\tau) - y_\tau^i(s^\tau)].$$

Therefore, $W_t^i(s^t)$ is the value of tail asset from the time-0 trading economy. It could differ from 0 for $t > 0$ because each household holds Arrow securities so that their future consumption could differ from sum of endowments. However, as Arrow securities only redistribute the consumption at each period,

$$\sum_{i=1}^{I} W_t^i(s^t) = 0.$$

In moving to the sequential formulation, we propose the state variable to match the value of tail asset from the time 0 trading economy. Let $\tilde{a}_t^i(s^t)$ denote the claims to time t consumption, other than its endowment, that household i brings into time t in history s_t. The household then faces a sequence of budget constraint,

$$\tilde{c}_t^i(s^t) + \sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) Q(s_{t+1}|s^t) = y_t^i(s^t) + \tilde{a}_t^i(s^t) \quad \forall t\,.$$

There is a number of s_{t+1} markets in one-period-ahead state-contingent claims to consumption. At each date $t \geq 0$, households trade claims to date $t+1$ consumption, whose payment is contingent on the realization of s_{t+1}. The pricing kernel $Q(s_{t+1}|s^t)$ gives the price of one unit of time $t+1$ consumption, contingent on the realization s_{t+1} at $t+1$, when the history at t is s^t.

Since we want to match the $\tilde{a}_t^i(s^t)$ to $W_t^i(s^t)$, the minimum $\tilde{a}_t^i(s^t)$ at each period is given by the natural debt limit

$$A_t^i(s^t) = -\sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} q_\tau^t(s^\tau) y_\tau^i(s^\tau).$$

Therefore, we also need to impose the following borrowing constraint on the sequential trading arrangement

$$-\tilde{a}^i_{t+1}(s^{t+1}) \leq A^i_{t+1}(s^{t+1}).$$

The household's problem

$$\begin{aligned}
&\max_{c^i_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u_i(c^i_t) \\
&s.t.\text{(Budget Constraint)} \quad \tilde{c}^i_t(s^t) + \sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t) Q(s_{t+1}|s^t) = y^i_t(s^t) + \tilde{a}^i_t(s^t) \quad \forall s^t, t, \text{ given } \tilde{a}^i_0(s_0) \\
&\quad \text{(Borrowing Constraint)} \quad -\tilde{a}^i_{t+1}(s^{t+1}) \leq A^i_{t+1}(s^{t+1}) \quad \forall s^{t+1} \\
&\quad \text{(Market Clearing)} \quad \sum_{i=1}^{I} \tilde{c}^i_t(s^t) = \sum_{i=1}^{I} y^i_t(s^t) \quad \forall s^t \\
&\quad \text{(Market Clearing)} \quad \sum_{i=1}^{I} \tilde{a}^i_{t+1}(s_{t+1}, s^t) = 0 \quad \forall s_{t+1}.
\end{aligned}$$

Lagrangian:

$$\begin{aligned}
\mathcal{L}^i = \sum_{t=0}^{\infty} \sum_{s^t} \quad & \beta^t u_i(\tilde{c}^i_t(s^t)) \pi_t(s^t) + \eta^i_t(s^t) \left[y^i_t(s^t) + \tilde{a}^i_t(s^t) - \tilde{c}^i_t(s^t) - \sum_{s_{t+1}} \tilde{a}^i_{t+1}(s_{t+1}, s^t) Q(s_{t+1}|s^t) \right] \\
& + \sum_{s_{t+1}} \nu^i_t(s_{t+1}, s^t)[A^i_{t+1}(s^{t+1}) + \tilde{a}^i_{t+1}(s^{t+1})]
\end{aligned}$$

First order condition:

$$\begin{aligned}
&[\tilde{c}^t_i(s^t)]: \quad \beta^t u'_i(\tilde{c}^i_t(s^t)) \pi_t(s^t) = \eta^i_t(s^t) \quad \forall t, s^t \\
&[\tilde{a}^{t+1}_i(s_{t+1}, s^t)]: \quad \eta^i_t(s^t) Q(s_{t+1}|s^t) = \eta^i_{t+1}(s_{t+1}, s^t) + \nu^i_t(s_{t+1}, s^t) \quad \forall t, s^t, s_{t+1}
\end{aligned}$$

In the optimal solution to this problem, the borrowing limit will not be binding, and hence the Lagrange multipliers $\nu^i_t(s_{t+1}, s^t)$ are all equal to zero. Therefore, we get the following equation,

$$Q_t(s_{t+1}|s^t) = \beta \frac{u'_i(\tilde{c}^i_{t+1}(s^{t+1}))}{u'_i(\tilde{c}^i_t(s^t))} \pi_t(s_{t+1}|s^t) \quad \forall t, s^t, s_{t+1}$$

6.2. Competitive equilibrium

Definition 6.1. *A sequential-trading competitive equilibrium is an initial distribution of wealth $\tilde{a}^i_0(s_0)$, an allocation $\{\tilde{c}^i_t(s^t)\}$, and pricing kernels $Q_t(s_{t+1}|s^t)$ such that*

- *$\forall i$, given $\tilde{a}^i_0(s_0)$ and $Q_t(s_{t+1}|s^t)$, the consumption allocation $\tilde{c}^i_t(s^t)$ solves the household's problem;*

- *$\forall s^t$, the households' consumption allocation and the implied asset portfolio $\{\tilde{c}_t^i(s^t), \{\tilde{a}_{t+1}^i\}_{s_{t+1}}\}$ satisfy the market clearing conditions $\sum_{i=1}^I \tilde{c}_t^i(s^t) = \sum_{i=1}^I y_t^i(s^t)$ and $\sum_{i=1}^I \tilde{a}_{t+1}^i(s_{t+1}, s^t) = 0$ for all s_{t+1}.*

We are going to prove that

1. Given $Q_t(s_{t+1}|s^t)$ to be $Q_t(s_{t+1}|s^t) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}$, where $q_t^0(s^t)$ is the state price in the time-0 trading economy, the equilibrium allocation of the time-0 trading economy is also the sequential competitive equilibrium.
   This can be shown by using the first-order condition for the time-0 trading economy $\beta^t u_i'(c_t^i(s^t))\pi_t(s^t) = \mu_i q_t^0(s^t)$, from which we get
   $$Q_t(s_{t+1}|s^t) = \beta \frac{u_i'(c_{t+1}^i(s^{t+1}))}{u_i'(c_t^i(s^t))}\pi_t(s_{t+1}|s^t).$$
   This is exactly the first-order condition for the sequential economy. It remains for us to choose the initial wealth of the sequential-trading equilibrium so that $\tilde{c}_t^i(s^t) = c_t^i(s^t)$. From the budget constraint, we can get that
   $$\sum_{s_{t+1}} \tilde{a}_{t+1}^i(s_{t+1}, s^t) q_{t+1}^0(s^{t+1}) = [y_t^i(s^t) - \tilde{c}_t^i(s^t)] q_t^0(s^t) + \tilde{a}_t^i(s^t) q_t^0(s^t)$$
   $$\sum_{s^{t+1}, s_{t+2}} \tilde{a}_{t+2}^i(s_{t+2}, s^{t+1}) q_{t+2}^0(s^{t+2}) = \sum_{s^{t+1}} [y_{t+1}^i(s^{t+1}) - \tilde{c}_{t+1}^i(s^{t+1})] q_{t+1}^0(s^{t+1}) + \sum_{s^{t+1}} \tilde{a}_{t+1}^i(s_{t+1}) q_{t+1}^0(s^{t+1})$$
   $$\sum_{s^{t+2}, s_{t+3}} \tilde{a}_{t+3}^i(s_{t+3}, s^{t+2}) q_{t+3}^0(s^{t+3}) = \sum_{s^{t+2}} [y_{t+2}^i(s^{t+2}) - \tilde{c}_{t+2}^i(s^{t+2})] q_{t+2}^0(s^{t+2}) + \sum_{s^{t+2}} \tilde{a}_{t+2}^i(s_{t+2}) q_{t+2}^0(s^{t+2})$$
   $$\cdots$$
   Adding the above equations all the way to infinity, we can get that
   $$\tilde{a}_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{s^\tau | s^t} [y_\tau^i(s^\tau) - \tilde{c}_\tau^i(s^\tau)] \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)}.$$
   As can be seen, if $\tilde{c}_t^i(s^t) = c_t^i(s^t)$, we have $\tilde{a}_t^i(s^t) = W_t^i(s^t)$. In this case, the market clearing conditions will be satisfied. Also, $\tilde{a}_0^i(s_0) = 0$, i.e. the initial wealth of all agents is zero. So the sequential-trading competitive equilibrium duplicates the Arrow-Debreu competitive equilibrium allocation.
2. The borrowing constraint precludes household from further increasing current consumption by reducing some component of the asset portfolio.
   If the household wants to ensure that consumption plan can be attained starting next period in all possible future states, the household should subtract the value of this commitment to

future consumption from the natural debt limit

$$-\tilde{a}^i_{t+1}(s^{t+1}) \leq A^i_{t+1}(s^{t+1}) - \sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^{t+1}} q^{t+1}_\tau(s^\tau) c^i_\tau(s^\tau) = -W^i_{t+1}(s^{t+1}) \quad \forall s^{t+1}$$

Hence, household i does not want to increase consumption at time t by reducing next period's wealth below $W^i_{t+1}(s^{t+1})$.

7. Recursive formulation

In the sequential trading economy, the findings hold for arbitrary individual endowment processes. At this level of generality, both the pricing kernels and the wealth distributions depend on the history s^t, which makes it extremely difficult to formulate an economic model that can be used to confront empirical observations. What we want is a framework where economic outcomes are functions of a limited number of "state variables" that summarize the effects of past events and current information. This desire leads us to make the following specialization of the exogenous forcing processes that facilitate a recursive formulation of the sequential-trading equilibrium.

If the endowments are governed by a Markov process, $\Pr(s_{t+1} = s'|s_t = s) = \pi(s'|s)$,

$$\pi_t(s^t) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})...\pi(s_1|s_0)\pi_0(s_0).$$

$\pi(s_0)$ is the initial distribution. Typically, we assumed that the trading begins after s_0 has been observed, which is captured by setting $\pi(s_0) = 1$ for any given value of s_0. Because of the Markov property, the conditional probability $\pi(s^t|s^\tau)$ $(t > \tau)$ depends only on the state s_τ at time τ and does not depend on the history before τ,

$$\pi_t(s^t|s^\tau) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2})...\pi(s_{\tau+1}|s_\tau).$$

Next, assume that the endowments are history independent

$$y^i_t(s^t) = y^i(s_t).$$

It follows immediately from the equilibrium allocation (history independence) that

$$c^i_t(s^t) = \bar{c}^i(s_t).$$

The pricing kernel is then

$$Q_t(s_{t+1}|s^t) = \beta \frac{u'_i(\bar{c}^i(s_{t+1}))}{u'_i(\bar{c}^i(s_t))} \pi(s_{t+1}|s_t) = Q(s_{t+1}|s_t).$$

Similarly, we can establish the history dependence of the wealth and the natural debt limit

$$\begin{aligned} W_t^i(s^t) &= \bar{W}^i(s_t) \\ A_t^i(s^t) &= \bar{A}^i(s_t) \end{aligned} \tag{2}$$

The above results enable us to formulate the households problem as a infinite horizon dynamic programming problem. The associated Bellman equation is given by

$$\begin{aligned} V^i(a^i, s) = \max_{c^i, \hat{a}^i} \quad & u_i(c^i) + \beta \sum_{s'} V^i(a'^i, s')\pi(s'|s) \\ s.t.\text{(Budget Constraint)} \quad & c_t^i + \sum_{s_{t+1}} \hat{a}_t^i(s_{t+1})Q(s_{t+1}|s_t) = y_t^i + a_t^i \quad \forall t, \\ \text{(Borrowing Constraint)} \quad & -\hat{a}_{t+1}^i(s_{t+1}) \leq \bar{A}_{t+1}^i(s_{t+1}) \quad \forall s_{t+1} \\ \text{(transition equation)} \quad & a_{t+1}^i = \hat{a}_t^i \quad \text{given } a_0^i, \\ \text{(Market Clearing)} \quad & \sum_{i=1}^I c_t^i = \sum_{i=1}^I y_t^i \\ \text{(Market Clearing)} \quad & \sum_{i=1}^I \hat{a}_{t+1}^i(s_{t+1}) = 0 \quad \forall s_{t+1}. \end{aligned} \tag{3}$$

The optimal policy is

$$\begin{aligned} c_t^i &= h^i(a_t^i, s_t) \\ \hat{a}_t^i(s) &= g_s^i(a_t^i, s_t) \end{aligned} \tag{4}$$

The envelope condition:

$$\frac{\partial V^i(a^i, s)}{\partial a^i} = \frac{\partial u_i(a^i, \hat{a}^i)}{\partial a^i}.$$

The first order condition:

$$0 = \frac{\partial u_i(a^i, \hat{a}^i)}{\partial \hat{a}^i(s')} + \beta \frac{\partial V^i(a'^i, s')}{\partial a'^i} \pi(s'|s).$$

Because $\frac{\partial u_i(a^i,\hat{a}^i)}{\partial a^i} = \frac{\partial u_i(c^i)}{\partial c^i}$, $\frac{\partial u_i(a^i,\hat{a}^i)}{\partial \hat{a}^i(s')} = -\frac{\partial u_i(c^i)}{\partial c^i} Q(s'|s)$, the Euler equation could be written as

$$Q(s_{t+1}|s_t)u_i'(c_t^i) = \beta u_i'(c_{t+1}^i)\pi(s_{t+1}|s_t).$$

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