Dynamic stochastic game and macroeconomic equilibrium

Tianxiao Zheng
SAIF

1. Introduction

We have studied single agent problems. However, macro-economy consists of a large number of agents including individuals/households, firms and banks, governments, and central banks, etc. It is now to leave the single-agent maximization setups and embark on the journey of macroeconomics.

Generalizing the single-agent maximization setup to the multiple-agent setting naturally leads to what is known as dynamic stochastic game. Recall that the Markov decision process is a limiting case of stochastic game with only one player. In this sense, macroeconomics is nothing else but a dynamic stochastic game.

The refinement of Nash equilibrium in dynamic game is known as **sub-game perfect equilibrium**. A strategy profile is a subgame perfect equilibrium if it represents a Nash equilibrium of every subgame of the original game. As common in modern macroeconomics, players condition their own strategies only on the payoff-relevant states in each period. In this case, the subgame perfect equilibrium in dynamic games is a **Markov perfect equilibrium**. The concept of Markov perfect equilibrium was first introduced by Maskin and Tirole, 1988. A Markov perfect equilibrium is derived as the solution to a set of dynamic programming problems that each individual player solves.

2. Dynamic stochastic games

2.1. Elements of dynamic stochastic game

- Decision epochs: $\mathcal{T} = 0, 1, 2, \ldots, T$, $T \leq \infty$
- A set of n players $\mathcal{I}$;
- A state space S;
- For each player $i \in \mathcal{I}$, an action space $A^{(i)}$; $A = \times_{i=1}^{n} A^{(i)}$;
- A transition probability P from $S \times A$ to S. $P(s'|s,a)$ is the probability that the next state is in s' given the current state s and the current action profile a;
- a payoff function $u = (u^{(1)}, ..., u^{(n)})$ from $S \times A$ to $\mathbb{R}^{\mathcal{I}}$; $u^{(i)}$ is the payoff function to player i as a function of the state s and action profile a;
- discount factor $\delta^{(i)}$ of player i

The game is played as follows. At time 1, the initial state $s_1 \in S$ is given. After observing the initial state, players choose their actions $a_1 \in A$ simultaneously and independently from each other. (In an equilibrium of a Stackelberg game, the timing of moves is so altered relative to the present game that one of the agents called the leader moves first and takes into account the influence that his choices exert on the other agent's choices.) Player i receives a payoff $u^{(i)}(s_1, a_1)$. Then the state evolves to s_2 according to the probability $P(s_2|s_1, a_1)$. After observing s_2, players choose their action $a_2 \in A$. Then players receive payoff $u^{(i)}(s_2, a_2)$, and the state of the system evolves again. The game continues in this way. Note that repeated games are special case of dynamic stochastic game with no state variables involved.

A general pure strategy for player i is a sequence $\sigma^{(i)} = \{\sigma_t^{(i)}\}_{t=1}^T$. $\sigma_t^{(i)}$ specifies a pure action to be taken at date t as a function of the history of all states up to date t. If it is history independent, then the strategy is said to be Markovian. For infinite-horizon games, if the payoff function and the transition probability is time-invariant, then the strategy is stationary. Let $\sigma = (\sigma^{(1)}, \sigma^{(2)}, ..., \sigma^{(n)})$ denote a Markov strategy profile.

The expected discounted payoff from a stationary Markov strategy is given by

$$\mathcal{U}^{(i)}(s_1; \sigma) = (1 - \delta^{(i)})\mathbb{E}\left[\sum_{t=1}^{\infty} (\delta^{(i)})^{t-1} u^{(i)}(s_t, \sigma_t(s_t)) \middle| s_1\right], \quad i = 1, 2, ..., n$$

2.2. Markov perfect equilibrium

Definition: A Markov strategy σ^* is a Markov perfect equilibrium if at any state s_t and any history s^t

$$\mathcal{U}^{(i)}(s_t; \sigma^*|s^t) \geq \mathcal{U}^{(i)}(s_t; \{\sigma_i, \sigma_{-i}^*\}|s^t), \qquad \text{for all } i \text{ and any } \sigma_i.$$

It is a special case of subgame perfect equilibrium.

Example 2.1. *The model consists of n firms selling output at each period. Firm i chooses a production plan $(y_{i,t})$ to maximize discounted profit*

$$\sum_{t=0}^{\infty} \beta_i^t u_{i,t} = \sum_{t=0}^{\infty} \beta_i^t \left[p_t y_{i,t} - 0.5 d_i (y_{i,t+1} - y_{i,t})^2\right]$$

subject to $y_{i,0}$ being a given initial condition. Here $\beta_i \in (0, 1)$ is a discount factor, and $d_i > 0$ measures a cost of adjusting the rate of output. The adjustment costs give the firm the incentive to forecast the market price p_t. The market price p_t lies in the inverse demand line

$$p_t = A_0 - A_1(y_{i,t} + y_{-i,t}),$$

where $y_{-i,t}$ denotes the output of the firm other than i. Here, we encounter a typical feature in a dynamic game: some quantity that one agent takes as exogenous are determined by all agents in the market. Specific to this model, p_t is determined by all the producers in the market.

Setting the control variable for firm i to be $a_{i,t} = y_{i,t+1}$, the transition equation is then $y_{i,t+1} = a_{i,t}$. Firm i chooses a policy that sets $a_{i,t}$ as a function of $(y_{i,t}, y_{-i,t})$. Then associated Bellman equation of firm i is

$$V_i(y_{i,t}, y_{-i,t}) = \underset{a_{i,t}}{max} \quad u_{i,t}(y_{i,t}, a_{i,t}) + \beta_i V_i(y_{i,t+1}, y_{-i,t+1}).$$

We see that to solve the optimization problem of firm i, we need all other firms' strategies $a_{j,t} = f_j(y_{j,t}, y_{-j,t})$. A Markov perfect equilibrium is a set of sequences $\{a_{1,t}, a_{2,t}, ..., a_{i,t}, ..., a_{n,t}\}$ such that $\{a_{1,t}\}$ solves the above Bellman equation for $i = 1$ given $\{a_{2,t}, ..., a_{n,t}\}$, $\{a_{2,t}\}$ solves the above Bellman equation for $i = 2$ given $\{a_{1,t}, a_{3,t}..., a_{n,t}\}$, and so on. Each firm's strategy depends only on the state variable $\vec{y}_t = [y_{1,t}, y_{2,t}, ..., y_{n,t}]'$.

In general, solving a dynamic game (a set of n interrelated Bellman equations to be solved by simultaneously backward induction) is hard. Below, we will introduce some sensible assumptions, rendering the Markov perfect equilibrium easily tractable.

3. Competitive equilibrium

We make the following assumption about the *population structure* of 2.1

- There is a large (or infinite) number of firms.

As a result, the output of any single firm has a negligible effect on the aggregate output. (no dominant player) Each firm is a price taker. This will lead to huge simplification of solving the above example.

The Bellman equation for firm i is now

$$\begin{aligned} V_i(y_{i,t}, y_{-i,t}) &= \max_{a_{i,t}} \quad A_0 y_{i,t} - A_1 y_{i,t}^2 - A_1 y_{i,t} y_{-i,t} - 0.5 d_i (a_{i,t} - y_{i,t})^2 + \beta_i V_i(y_{i,t+1}, y_{-i,t+1}) \\ V_i(y_{i,t}, p_t) &\approx \max_{a_{i,t}} \quad A_0 y_{i,t} - A_1 y_{i,t}^2 - A_1 y_{i,t} y_{-i,t} - 0.5 d_i (a_{i,t} - y_{i,t})^2 + \beta_i V_i(y_{i,t+1}, p_{t+1}) \end{aligned} \tag{1}$$

where $p_{t+1} \approx A_0 - A_1 y_{-i,t+1}$. If given the price p_{t+1}, we could solve for the optimal policy $a_{i,t}$ as a function of p_{t+1} and the state variable $y_{i,t}$. Plugging $a_{i,t}$ into $p_{t+1} = A_0 - A_1 \sum_j a_{j,t}$, we could solve for p_{t+1}.

Definition 3.1. *Competitive equilibrium*

Competitive equilibrium is usually defined as given the price that equates demand and supply.

4. Recursive competitive equilibrium

Representative agent is a very useful concept in formalizing many economic problems in a parsimonious way. It allows to define the competitive equilibrium recursively. We make the following assumptions about the *population structure* of 2.1

- There is a large (or infinite) number of firms.
- Every firm is identical.

Note that the assumption of each firm being identical amounts to dropping the distribution of output size.

The Bellman equation for firm i is now

$$\begin{aligned} V_i(y_{i,t}, y_{-i,t}) &= \max_{a_{i,t}} \quad A_0 y_{i,t} - A_1 y_{i,t}^2 - A_1 y_{i,t} y_{-i,t} - 0.5 d_i (y_{i,t+1} - y_{i,t})^2 + \beta_i V_i(y_{i,t+1}, y_{-i,t+1}) \\ V_i(y_{i,t}, Y_t) &\approx \max_{a_{i,t}} \quad A_0 y_{i,t} - A_1 y_{i,t}^2 - A_1 y_{i,t} Y_t - 0.5 d_i (a_{i,t} - y_{i,t})^2 + \beta_i V_i(y_{i,t+1}, Y_{t+1}) \end{aligned} \tag{2}$$

where $Y_t = \sum_{i=1}^n y_{i,t} \approx y_{-i,t}$ is the total output by all firms. We see that to solve the optimization problem of firm i, we need the law of motion for Y_t

$$Y_{t+1} = H(Y_t).$$

Since every firm is identical, we could drop the subscript i in the Bellman equation. Based on the above discussion, we could come up with a recursive way to find the equilibrium of the model,

1. given H, find the optimal policy of the Bellman equation, $a_t = y_{t+1} = h(y_t, Y_t)$.
2. form the law of motion for Y_t according to the optimal policy $Y_{t+1} = nh(Y_t/n, Y_t) = H'(Y_t)$.
3. given H', repeat step (1) and (2) until convergence is attained.

The firm's optimum problem induces a mapping $\mathcal{M}$ from a perceived law of motion for Y_t, H to an actual law of motion $\mathcal{M}(H)$. H is a fixed point of the operator $\mathcal{M}$. A recursive competitive equilibrium equates the actual and perceived laws of motion.

For convenience, we usually set the number of firms n to be 1. In the second step of the above procedure, we impose $Y_t = y_t$ so that $Y_{t+1} = h(Y_t)$. Note that this is only after we have solved the representative agent's decision problem.

Definition 4.1. *Recursive competitive equilibrium of infinite horizon optimal control problem.*

Let X be the vector of x of the market. The representative agent's problem is characterized by the Bellman equation

$$V(x, z) = \max_{a \in \Gamma(x,z)} u(x, X, a, z) + \beta \mathbb{E}_t V(x', X', z'),$$

with the transition equations

$$\begin{aligned} x' &= \phi(x, X, a, z), \\ X' &= \Phi(X, z), \\ z' &= \zeta(z). \end{aligned} \tag{3}$$

The optimal policy of the representative agent's problem is

$$a = g(x, X, z)$$

Substituting this equation into the transition equation of x yields

$$X' = n\phi(X/n, X, g(X/n, X, z), z) = \phi(X, X, g(X, X, z), z) = \Phi_a(X, z).$$

A recursive competitive equilibrium is a policy function g, an actual aggregate law of motion Φ_a, and a perceived aggregate law Φ such that (a) given Φ, g solves the representative agent's optimization problem; and (b) g implies that $\Phi_a = \Phi$.

Recursive competitive equilibrium is also sometimes called a rational expectations equilibrium. The equilibrium concept makes Φ an outcome of the analysis. The functions giving the representative agent's expectations about the aggregate state variables contribute no free parameters and are outcomes of the analysis. There are no free parameters that characterize expectations.

Example 4.1. *The economy consists of a firm and a large number of n households. The firm faces a wage process w_t and chooses a plan for hired labor L_t to maximize,*

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{f_0 + (f_1 + \theta_t)L_t - f_2 L_t^2/2 - w_t L_t\},$$

subject to the transition equation $\theta_{t+1} = \rho\theta_t + \sigma\epsilon_{t+1}$. A representative household chooses an amount of labor (s_t) to send to school that takes four periods to produce an educated worker. It maximizes

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(w_t l_t - \frac{d}{2} s_t^2 \right),$$

subject to the law of motion $l_{t+1} = \delta l_t + s_{t-3}$.

- *The state of the economy is given by $\{\theta_t, l_{t+3}\}$.*
- *The policy of the firm is simple (Note that in this example, we do not need the policy of households in order to solve the problem of the firm.)*

$$f_1 + \theta_t - f_2 L_t = w_t.$$

  This could be viewed as the inverse demand line for the stock of labor $L_t = n l_t$.
- *Given the policy of the firm, we maximize the value function of the household*

$$\mathbb{E}_0 \beta^3 \sum_{t=0}^{\infty} \beta^t \left(w_{t+3} x_t - \frac{d}{2\beta^3} s_t^2 \right),$$

  where $x_t = l_{t+3}$ is the state variable. The transition equation is now $x_{t+1} = \delta x_t + s_t$. By *setting $x_{t+1} = a_t$, we have $s_t = a_t - \delta x_t$. The maximization problem becomes*

$$\max_{a_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(w_{t+3} x_t - \frac{d}{2\beta^3} (a_t - \delta x_t)^2 \right),$$

subject to $x_{t+1} = a_t$. The Euler equation is given by

$$\frac{d}{\beta^3} s_t = \beta \mathbb{E}_t \left[w_{t+4} + \frac{d\delta}{\beta^3} s_{t+1} \right].$$

Using the method of lag operator, the Euler equation can be written as $(1 - \delta\beta\mathcal{L}^{-1})s_t = d^{-1}\beta^4 \mathbb{E}_t w_{t+4}$. As a result,

$$s_t = d^{-1}\beta^4 \sum_{j=0}^{\infty} (\delta\beta)^j \mathcal{L}^{-j} \mathbb{E}_t w_{t+4} = d^{-1}\beta^4 \sum_{j=0}^{\infty} (\delta\beta)^j \mathbb{E}_t w_{t+j+4}.$$

The law of motion is therefore, $l_{t+1} = \delta l_t + s_{t-3}$.

$$l_{t+4} = \delta l_{t+3} + d^{-1}\beta^4 \sum_{j=0}^{\infty} (\delta\beta)^j \mathbb{E}_t w_{t+j+4}.$$

- *Recursive competitive equilibrium of households*
  the big K, little k trick gives

$$L_{t+4} = \delta L_{t+3} + d^{-1}\beta^4 \sum_{j=0}^{\infty} (\delta\beta)^j \mathbb{E}_t w_{t+j+4}.$$

References

Maskin, E., Tirole, J., 1988. A theory of dynamic oligopoly, i: Overview and quantity competition with large fixed costs. Econometrica 56, 549–569.