Linear rational expectation system

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1. Introduction

Solving the infinite-horizon optimal control problem often gives rise to the following first-order nonlinear system

$$\begin{aligned} 0 &= u_a(x_t, z_t, a_t) + \beta \mathbb{E}_t \mu_{t+1} \phi_a(x_t, z_t, a_t, z_{t+1}), \quad t > 0 \\ \mu_t &= u_x(x_t, z_t, a_t) + \beta \mathbb{E}_t \mu_{t+1} \phi_x(x_t, a_t, z_t, z_{t+1}), \quad t \geq 1 \\ x_{t+1} &= \phi(x_t, a_t, z_t, z_{t+1}), \quad x_0 \text{ given, } t \geq 0 \\ z_{t+1} &= \Phi z_t + \Sigma \epsilon_{t+1}, \quad z_0 \text{ given, } t \geq 0 \end{aligned} \tag{1}$$

$\{z_t\}$ is a bounded $\mathbb{R}^{n_z}$-valued stochastic process, Φ is an $n_z \times n_z$ matrix, Σ is an $n_z \times n_\epsilon$ matrix, and $\{\epsilon_{t+1}\}$ is a stochastic process satisfying $\mathbb{E}_t \epsilon_{t+1} = 0$ and $\mathbb{E} \epsilon_{t+1} \epsilon'_{t+1} = I$.

Many macroeconomic problems can be described by the above system. As we possess a kaleidoscope of tools to deal with linear system, the first-order nonlinear system is usually linearized around its non-deterministic steady state. The resulting linear system is then solved and analyzed. [1]

Example 1.1. *Consider the following problem as an example*

$$\max_{\{C_t, K_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \log C_t$$

subject to

$$K_{t+1} = Z_t K_t^{\alpha} - C_t$$

where β, $\alpha \in (0,1)$ and Z_t satisfies

$$\log(Z_{t+1}) = \rho \log(Z_t) + \sigma \varepsilon_{t+1}$$

[1] For higher order treatments, see for example (1)`http://www.columbia.edu/~mu2166/2nd_order.htm`; (2) `http://www.nber.org/papers/w18983`

Solving the problem yields the following set of system equation,

$$\begin{aligned} K_{t+1} &= Z_t K_t^\alpha - C_t \\ \frac{1}{C_t} &= \alpha\beta\mathbb{E}_t[\frac{1}{C_{t+1}} Z_{t+1} K_{t+1}^{\alpha-1}] \end{aligned}$$

Deterministic steady state satisfies the following equation (in the deterministic case, $Z_t = 1$):

$$\begin{aligned} \bar{K} &= \bar{K}^\alpha - \bar{C} \\ 1 &= \alpha\beta\bar{K}^{\alpha-1} \end{aligned}$$

Therefore, $\bar{K} = (\alpha\beta)^{1/(1-\alpha)}$, and $\bar{C} = \bar{K}(\bar{K}^{\alpha-1} - 1) = (\alpha\beta)^{1/(1-\alpha)}(1/(\alpha\beta) - 1)$.

Defining $K_t = \bar{K}\exp(\hat{k}_t)$ and $C_t = \bar{C}\exp(\hat{c}_t)$, where $\hat{k}_t$ and $\hat{c}_t$ are the percentage deviations of capital and consumption around their deterministic steady state, and plugging them into system equations, we obtain

$$\begin{aligned} \bar{K}\exp(\hat{k}_{t+1}) &= \bar{Z}\exp(\hat{z}_t)\bar{K}^\alpha \exp(\hat{k}_t\alpha) - \bar{C}\exp(\hat{c}_t) \\ \frac{1}{\bar{C}\exp(\hat{c}_t)} &= \alpha\beta\mathbb{E}_t[\frac{1}{\bar{C}\exp(\hat{c}_{t+1})}\bar{Z}\exp(\hat{z}_{t+1})\bar{K}^{\alpha-1}\exp(\hat{k}_{t+1}(\alpha-1))] \end{aligned}$$

Here, we define $Z_t = \bar{Z}\exp(\hat{z}_t)$, where $\bar{Z}$ is the steady state mean of Z_t. Because $\log(Z_{t+1}) = \rho\log(Z_t) + \sigma\varepsilon_{t+1}$*. Therefore, $\bar{Z} = 1$ and $\hat{z}_{t+1} = \rho\hat{z}_t + \sigma\varepsilon_{t+1}$. Using the first Taylor expansion,* $\exp(x) \approx 1 + x$ *for x near 0, we have*

$$\begin{gathered} \hat{k}_{t+1} = \frac{1}{\beta}\hat{k}_t + (1 - \frac{1}{\alpha\beta})\hat{c}_t + \frac{1}{\alpha\beta}\hat{z}_t \\ \mathbb{E}_t[(1-\alpha)\hat{k}_{t+1} + \hat{c}_{t+1}] = \hat{c}_t + \rho\hat{z}_t \end{gathered}$$

Defining $x_t = [\hat{k}_t, \hat{c}_t]'$, we get

$$A\mathbb{E}_t x_{t+1} = Bx_t + Cz_t$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 1-\alpha & 1 \end{bmatrix}, \quad B = \begin{bmatrix} \beta^{-1} & 1-(\alpha\beta)^{-1} \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} (\alpha\beta)^{-1} \\ \rho \end{bmatrix}$$

In this lecture, we introduce the method to solve linear rational expectations models:

$$A\mathbb{E}_t x_{t+1} = Bx_t + Cz_t\,, \tag{2}$$

Here, $x_t \in \mathbb{R}^n$ is a random vector, A and B are $n \times n$ matrices, and C is a $n \times n_z$ matrix.

2. Scalar Equation

In this section, we consider the scalar first-order equation ($n = n_z = 1$ in Eq. 2)

$$\mathbb{E}_t x_{t+1} = b x_t + c z_t \tag{3}$$

For example, the asset-pricing equation is of form Eq. 3, $p_t = \frac{\mathbb{E}_t p_{t+1} + d_t}{R}$, where $R > 1$ and d_t satisfies $d_t = \rho d_{t-1} + \sigma \epsilon_t$.

Iterating Eq. 3 backward, we obtain the backward looking solution:

$$x_t = b^t x_0 + c \sum_{j=0}^{t-1} b^j z_{t-j-1}$$

Iterating Eq. 3 forward, we obtain the forward looking solution

$$x_t = b^{-T} \mathbb{E}_t x_{t+T} - c \sum_{j=0}^{T-1} b^{-j-1} \mathbb{E}_t z_{t+j}$$

In general, the solution to Eq. 3 is given by $x_t = x_t^h + x_t^p$, where x_t^p is a particular solution to (2) and x_t^h is the solution to the homogeneous equation $\mathbb{E}_t x_{t+1}^h = b x_t^h$. To find x_t^p, we introduce the widely adopted method: the lag operator method. The lag operator on any stochastic process $\{X_t\}$ is defined by

$$\mathcal{L}^j X_t = X_{t-j}, \quad \mathcal{L}^{-j} X_t = \mathbb{E}_t X_{t+j}, \quad j > 0\,. \tag{4}$$

Using the inverse of lag operator, Eq. 3 can be rewritten as

$$(\mathcal{L}^{-1} - b) x_t = c z_t.$$

There exist two forms of particular solutions x_t^p:

- forward looking solution

$$x_t^p = -\frac{c}{b} \frac{1}{1 - b^{-1}\mathcal{L}^{-1}} z_t = -c \sum_{j=0}^{\infty} b^{-j-1} \mathcal{L}^{-j} z_t = -c \sum_{j=0}^{\infty} b^{-j-1} \mathbb{E}_t z_{t+j}$$

  if $|b| > 1$, then the infinite sum above is finite since z_t is a bounded sequence.
- backward looking solution

$$x_t^p = \frac{c}{\mathcal{L}^{-1}} \frac{1}{1 - b\mathcal{L}} z_t = c \sum_{j=0}^{\infty} b^j \mathcal{L}^{j+1} z_t = c \sum_{j=0}^{\infty} b^j z_{t-j-1}$$

  if $|b| < 1$, then the infinite sum above is finite since z_t is a bounded sequence.

Now we proceed to find x_t^h satisfying $\mathbb{E}_t x_{t+1}^h = b x_t^h$. There also exist two forms of solution

- forward looking solution $x_t^h = b^{-T}\mathbb{E}_t x_{t+T}$
  Transversality condition (or no-bubble condition): $\lim_{T\to\infty} b^{-T}\mathbb{E}_t x_{t+T} = 0$
- backward looking solution $\mathbb{E}_0 x_t^h = b^t x_0$
  Initial condition x_0 should be given here, or else the solution is indeterminate.

If no transversality or initial condition is given, x_t^h admits many solutions so long as x_t^h satisfies $\mathbb{E}_t x_{t+1}^h = b x_t^h$.

In this example, we have shown that two conditions are important for solving a linear difference equation:

- whether the initial value is given;
- whether the coefficient b is smaller than one in absolute value.

We will show below that similar conditions apply to general multivariate linear systems. In particular, the first condition determines whether the variable x_t is *predetermined*, and the second condition corresponds to whether *the eigenvalue is stable.*

3. Blanchard-Kahn method

We focus on the case of invertible A where the method of Blanchard and Kahn, 1980 can be applied. (Klein, 2000 allows A to be singular, where generalized Schur form was employed) Eq. 2 can be rewritten as

$$\mathbb{E}_t x_{t+1} = A^{-1}Bx_t + A^{-1}Cz_t\,.$$

As usual, we proceed from here by diagonalizing $W = A^{-1}B$ first. However, in the general cases, it is only possible to represent W in the Jordan form such that $W = P^{-1}JP$, where J is a Jordan matrix:

$$J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \dots & \\ & & & J_l \end{bmatrix}.$$

The Jordan blocks J_i $(i = 1 \dots l)$ are matrices composed of 0 elements everywhere except for the diagonal, which is filled with a fixed element λ, and for the superdiagonal, which is composed of ones.

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \dots & 1 \\ & & & \lambda_i \end{bmatrix}.$$

Here, λ are the eigenvalues of W.

Defining $x_t^* = Px_t$ and $C^* = PA^{-1}C$

$$\mathbb{E}_t x_{t+1}^* = Jx_t^* + C^* z_t\,.$$

In economics, we are most interested in the saddle path solution to the above equation, where the modulus of some λ_i is greater than 1 and the others less than 1. As will be shown later, this has to do with the fact that initial value of some components of x_t are not exogenously given, or *non-predetermined*. We have ruled out the case when some of the eigenvalues are on the unit circle ($|\lambda_i| = 1$), because generally this leads to unstable behavior. As $|\lambda_i| > 1$ leads to explosive behavior, we partition the system into two parts: $|\lambda_i| > 1$ (n_u of them) and $|\lambda_i| < 1$ ($n_s = n - n_u$ of them). Jordan matrix J is partitioned as

$$J = \begin{bmatrix} J_s & \\ & J_u \end{bmatrix},$$

where J_s contains all $|\lambda_i| < 1$ and J_u all $|\lambda_i| > 1$.

x_t^* and C^* are partitioned accordingly,

$$x_t^* = \begin{bmatrix} s_t \\ u_t \end{bmatrix}, \quad C^* = \begin{bmatrix} C_s^* \\ C_u^* \end{bmatrix}.$$

As a result, the system can be written as

$$\begin{bmatrix} \mathbb{E}_t s_{t+1} \\ \mathbb{E}_t u_{t+1} \end{bmatrix} = \begin{bmatrix} J_s & 0 \\ 0 & J_u \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} + \begin{bmatrix} C_s^* \\ C_u^* \end{bmatrix} z_t \tag{5}$$

We first take care of u_t. Applying the lag operators, we have

$$\mathcal{L}^{-1} u_t = J_u u_t + C_u^* z_t.$$

The solution to u_t is straightforward

$$u_t = -(J_u - \mathcal{L}^{-1})^{-1} C_u^* z_t = -J_u^{-1}(1 - \mathcal{L}^{-1} J_u^{-1})^{-1} C_u^* z_t = -\sum_{j=0}^{\infty} J_u^{-j-1} C_u^* \mathbb{E}_t z_{t+j}\,. \tag{6}$$

Here, J_u contains all the $|\lambda_i| > 1$, so as long as z_t is a stable stochastic process the above sum series converges.

Transforming back to x_t (remember $x_t^* = P x_t$), we have

$$\begin{bmatrix} s_t \\ u_t \end{bmatrix} = P \begin{bmatrix} k_t \\ y_t \end{bmatrix} = \begin{bmatrix} P_{sk} & P_{sy} \\ P_{uk} & P_{uy} \end{bmatrix} \begin{bmatrix} k_t \\ y_t \end{bmatrix}. \tag{7}$$

We have partitioned x_t as $[k_t', y_t']'$, where vector y_t contains n_y non-predetermined components and k_t contains $n_k = n - n_y$ predetermined components. Therefore,

$$s_t = P_{sy} y_t + P_{sk} k_t, \tag{8}$$

$$u_t = P_{uy} y_t + P_{uk} k_t. \tag{9}$$

In particular, we consider

$$u_0 = P_{uy} y_0 + P_{uk} k_0$$

, where k_0 is predetermined, u_0 can be derived from the above equation. P_{uy} is an $n_u \times n_y$ matrix. If $n_u > n_y$, there can be no solution for y_0. If $n_u < n_y$, there can be infinitely many solutions for y_0. We therefore consider the case where the Blanchard-Kahn condition is satisfied:

- $n_y = n_u$, i.e. the number of unstable eigenvalues is equal to the number of non-predetermined variables. (forward-looking variables)
- P_{uy} is invertible.

Under the condition, we have

$$y_t = P_{uy}^{-1} u_t - P_{uy}^{-1} P_{uk} k_t \,. \tag{10}$$

To solve for k_t, we use the following definition

$$\begin{bmatrix} k_t \\ y_t \end{bmatrix} = P^{-1} \begin{bmatrix} s_t \\ u_t \end{bmatrix} = \begin{bmatrix} R_{ks} & R_{ku} \\ R_{ys} & R_{yu} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} , \tag{11}$$

to get

$$k_{t+1} = R_{ks} s_{t+1} + R_{ku} u_{t+1}$$

Note that $\mathbb{E}_t s_{t+1} = J_s s_t + C_s^* z_t$. According to Blanchard and Kahn, 1980, $\mathbb{E}_t k_{t+1} = k_{t+1}$. Therefore, we have $s_{t+1} = J_s s_t + C_s^* z_t$ and

$$\begin{aligned} k_{t+1} &= \mathbb{E}_t k_{t+1} = R_{ks}(J_s s_t + C_s^* z_t) + R_{ku}(J_u u_t + C_u^* z_t) \\ &= R_{ks}(J_s(P_{sk} k_t + P_{sy} y_t) + C_s^* z_t) + R_{ku}(J_u u_t + C_u^* z_t) \\ &= R_{ks}(J_s(P_{sk} k_t + P_{sy}(P_{uy}^{-1} u_t - P_{uy}^{-1} P_{uk} k_t)) + C_s^* z_t) + R_{ku}(J_u u_t + C_u^* z_t) \end{aligned} \tag{12}$$

Because $R_{ks}(P_{sk} - P_{sy} P_{uy}^{-1} P_{uk}) = I$,

$$k_{t+1} = R_{ks} J_s R_{ks}^{-1} k_t + (R_{ku} C_u^* + R_{ks} C_s^*) z_t + (R_{ku} J_u P_{uy} + R_{ks} J_s P_{sy}) P_{uy}^{-1} u_t. \tag{13}$$

Note that $R_{ks} J_s R_{ks}^{-1}$ has the same eigenvalues as J_s. So the evolution of k_t is stable. Eqs. 6, 10, and 13 form the solution to the problem of Eq. 2. If the z_t is given, the solution can be simplified,

$$\begin{aligned} y_t &= A_{yk} k_t + A_{yz} z_t \\ k_{t+1} &= A_{kk} k_t + A_{kz} z_t \\ z_{t+1} &= \Phi z_t + \Sigma \epsilon_{t+1} \end{aligned} \tag{14}$$

In Klein, 2000, predetermined variables are defined with an prediction error $\xi_{t+1} = k_{t+1} - \mathbb{E}_t k_{t+1}$. ξ_t is an exogenously given martingale difference process ($\mathbb{E}_t \xi_{t+1} = 0$). The structure of the solution is now

$$\begin{aligned} y_t &= A_{yk}k_t + A_{yz}z_t \\ k_{t+1} &= A_{kk}k_t + A_{kz}z_t + \xi_{t+1} \\ z_{t+1} &= \Phi z_t + \Sigma\epsilon_{t+1} \end{aligned} \tag{15}$$

Example 3.1. *Stochastic second-order equation*

$$\mathbb{E}_t x_{t+1} = a x_t + b x_{t-1} + c z_t$$

We first show that the above equation is of form Eq. 2.

$$\mathbb{E}_t \begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} 0 \\ c \end{bmatrix} z_t$$

Note that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

For illustration, we assign the values of a, b, c, $a = 2.5$, $b = -1$, *and* $c = 1$. *Remember that* $W = A^{-1}B$,

$$W = P^{-1}JP = \begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix}^{-1}$$

Therefore, $\lambda_1 = 1/2$, $\lambda_2 = 2$, [2]

$$P = \begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ -2/3 & 4/3 \end{bmatrix}, \quad J = \begin{bmatrix} J_s & 0 \\ 0 & J_u \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix}$$

Defining

$$x_t^* = \begin{bmatrix} s_t \\ u_t \end{bmatrix} = P \begin{bmatrix} k_t \\ y_t \end{bmatrix} = P \begin{bmatrix} x_{t-1} \\ x_t \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ -2/3 & 4/3 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ x_t \end{bmatrix}$$

we have (here, $k_t \equiv x_{t-1}$ *and* $y_t \equiv x_t$*)*

$$\begin{aligned} s_t &= \frac{2}{3}k_t - \frac{1}{3}y_t, \\ u_t &= -\frac{2}{3}k_t + \frac{4}{3}y_t. \end{aligned}$$

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[2] The two eigenvalues of W satisfy $\det(W - \lambda) = 0$, giving $\lambda_{1,2} = \frac{a}{2} \pm \sqrt{a^2/4 + b}$. By definition, the according eigenvectors e_i must satisfy the relationship $We_i = {}_ie_i$ for each eigenvalue. Standardizing the eigenvectors to $e_i = [1, x_i]^T$ allows one to obtain $e_{1,2} = [1, \lambda_{1,2}]^T$. Defining the matrix $E = [e_1, e_2]$, we have $WE = EJ$, where

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

With

$$C^* = PA^{-1}C = \begin{bmatrix} 2/3 & -1/3 \\ -2/3 & 4/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 4/3 \end{bmatrix},$$

we get

$$\mathbb{E}_t \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} J_s & 0 \\ 0 & J_u \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} + \begin{bmatrix} C_s^* \\ C_u^* \end{bmatrix} z_t = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} + \begin{bmatrix} -1/3 \\ 4/3 \end{bmatrix} z_t.$$

That is

$$\begin{aligned} \mathbb{E}_t s_{t+1} &= J_s s_t + C_s^* z_t = \frac{1}{2} s_t - \frac{1}{3} z_t \\ \mathbb{E}_t u_{t+1} &= J_u u_t + C_u^* z_t = 2u_t + \frac{4}{3} z_t \end{aligned}$$

Solving u_t using the method of lag operator,

$$\mathcal{L}^{-1} u_t = J_u u_t + C_u^* z_t = 2u_t + \frac{4}{3} z_t$$

we obtain

$$u_t = -\sum_{j=0}^{\infty} J_u^{-j-1} C_u^* \mathbb{E}_t z_{t+j} = -\sum_{j=0}^{\infty} 2^{-j-1} \frac{4}{3} \mathbb{E}_t z_{t+j}$$

Because $u_t = -\frac{2}{3} k_t + \frac{4}{3} y_t$,

$$y_t = \frac{3}{4} u_t + \frac{1}{2} k_t$$

Because

$$\begin{bmatrix} k_t \\ y_t \end{bmatrix} = P^{-1} \begin{bmatrix} s_t \\ u_t \end{bmatrix} = \begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} \tag{16}$$

we have

$$k_{t+1} = 2s_{t+1} + \frac{1}{2} u_{t+1}.$$

Note that $\mathbb{E}_t k_{t+1} = \mathbb{E}_t x_t = x_t = k_{t+1}$

$$\begin{aligned} k_{t+1} = \mathbb{E}_t k_{t+1} &= 2(J_s s_t + C_s^* z_t) + \frac{1}{2}(J_u u_t + C_u^* z_t) \\ &= 2\left(\frac{1}{2} s_t - \frac{1}{3} z_t\right) + \frac{1}{2}\left(2u_t + \frac{4}{3} z_t\right) \\ &= 2\left(\frac{1}{2}\left(\frac{2}{3} k_t - \frac{1}{3} y_t\right) - \frac{1}{3} z_t\right) + \frac{1}{2}\left(2u_t + \frac{4}{3} z_t\right) \\ &= 2\left(\frac{1}{2}\left(\frac{2}{3} k_t - \frac{1}{3}(\frac{3}{4} u_t + \frac{1}{2} k_t)\right) - \frac{1}{3} z_t\right) + \frac{1}{2}\left(2u_t + \frac{4}{3} z_t\right) \\ &= \frac{1}{2} k_t + \frac{3}{4} u_t \end{aligned} \tag{17}$$

As a result, we have $x_t = \frac{1}{2} x_{t-1} - \sum_{j=0}^{\infty} 2^{-j-1} \mathbb{E}_t z_{t+j}$.

Typically, $z_t = \rho z_{t-1} + \sigma \epsilon_t$. Therefore, $\mathbb{E}_t z_{t+1} = \rho z_t$, $\mathbb{E}_t z_{t+2} = \rho \mathbb{E}_t z_{t+1} = \rho^2 z_t$, $\mathbb{E}_t z_{t+3} =$

$\rho\mathbb{E}_t z_{t+2} = \rho^3 z_t$, ... *and so on. In this case,* $x_t = \frac{1}{2}x_{t-1} - \frac{1}{2}z_t \sum_{j=0}^{\infty}(\frac{1}{2})^j \rho^j = \frac{1}{2}x_{t-1} - \frac{z_t}{2-\rho}$.

4. Further reading

1. Chris Sims' website on solving linear RE models and his note on the computational method.
2. Harold Uhlig's website for "A toolkit for analyzing nonlinear economic dynamic models."

Appendix A. Solution to Eq. 3 with the method of undetermined coefficients

Uhlig (1999) ("A toolkit for analyzing nonlinear dynamic stochastic models easily" in *Computational Methods for the Study of Dynamic Economics*, Oxford University Press) provides a toolkit for solving linear and nonlinear systems using the method of undetermined coefficients.

Assume that $x_t = Gx_{t-1} + Hz_t$ and find G and H

$$\mathbb{E}_t(Gx_t + Hz_{t+1}) = a(Gx_{t-1} + Hz_t) + bx_{t-1} + cz_t = G^2x_{t-1} + GHz_t + \rho Hz_t$$

Comparing the coefficients in the above equation yields

$$\begin{aligned} aG + b &= G^2 \\ aH + c &= GH + \rho H \end{aligned} \tag{18}$$

With $a = 2.5$, $b = -1$, and $c = 1$, we have $G = 1/2$ or $G = 2$. To have a stable and non-exploding solution, we pick the solution with modulus less than 1, i.e. $G = 1/2$. The value of H is then $-\frac{1}{2-\rho}$.

Therefore, we once again get $x_t = \frac{1}{2}x_{t-1} - \frac{z_t}{2-\rho}$.

Appendix B. A third solution to Eq. 3

We rewrite the equation as

$$(\mathcal{L}^{-1} - a - b\mathcal{L})x_t = cz_t$$

Let λ_1 and λ_2 be the two characteristic roots,

$$(\mathcal{L}^{-1} - \lambda_1)(\mathcal{L}^{-1} - \lambda_2)\mathcal{L}x_t = cz_t$$

where $\lambda_1 + \lambda_2 = a$ and $\lambda_1\lambda_2 = -b$. In general, there are three cases

- λ_1 and λ_2 are real and distinctive
- λ_1 and λ_2 are the same
- λ_1 and λ_2 are complex conjugate.

We are most interested in the case $|\lambda_1| < 1$ and $|\lambda_2| > 1$.

$$(\mathcal{L}^{-1} - \lambda_1)\mathcal{L}x_t = -\frac{cz_t}{\lambda_2(1 - \lambda_2^{-1}\mathcal{L}^{-1})}$$

We then obtain the solution:

$$x_t = \lambda_1 x_{t-1} - \frac{c}{\lambda_2}\sum_{j=0}^{\infty}\lambda_2^{-j}\mathcal{L}^{-j}z_t = \lambda_1 x_{t-1} - \frac{c}{\lambda_2}\sum_{j=0}^{\infty}\lambda_2^{-j}\mathbb{E}_t z_{t+j}$$

Note that we need initial condition x_0 to complete the solution.

References

Blanchard, O. J., Kahn, C. M., 1980. The solution of linear difference models under rational expectations. Econometrica 48, 1305–1311.

Klein, P., 2000. Using the generalized schur form to solve a multivariate linear rational expectations model. Journal of Economic Dynamics and Control 24, 1405 – 1423.